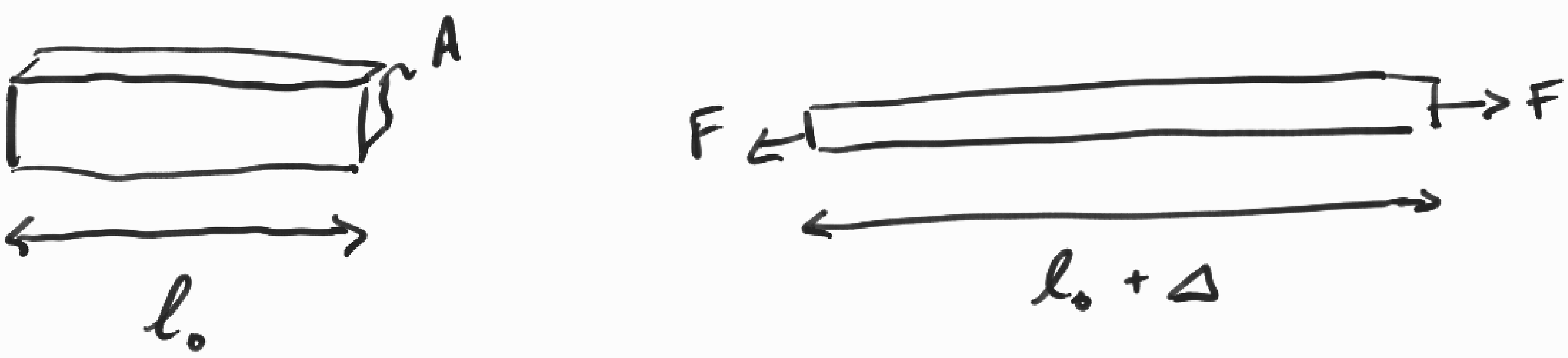
4. STRAIN & STRESS

CONSIDER UNIAXIAL EXTENSION (ALONG ONLY 1 DIRECTION) OF A BAR

UNDEFORMED   DEFORMED



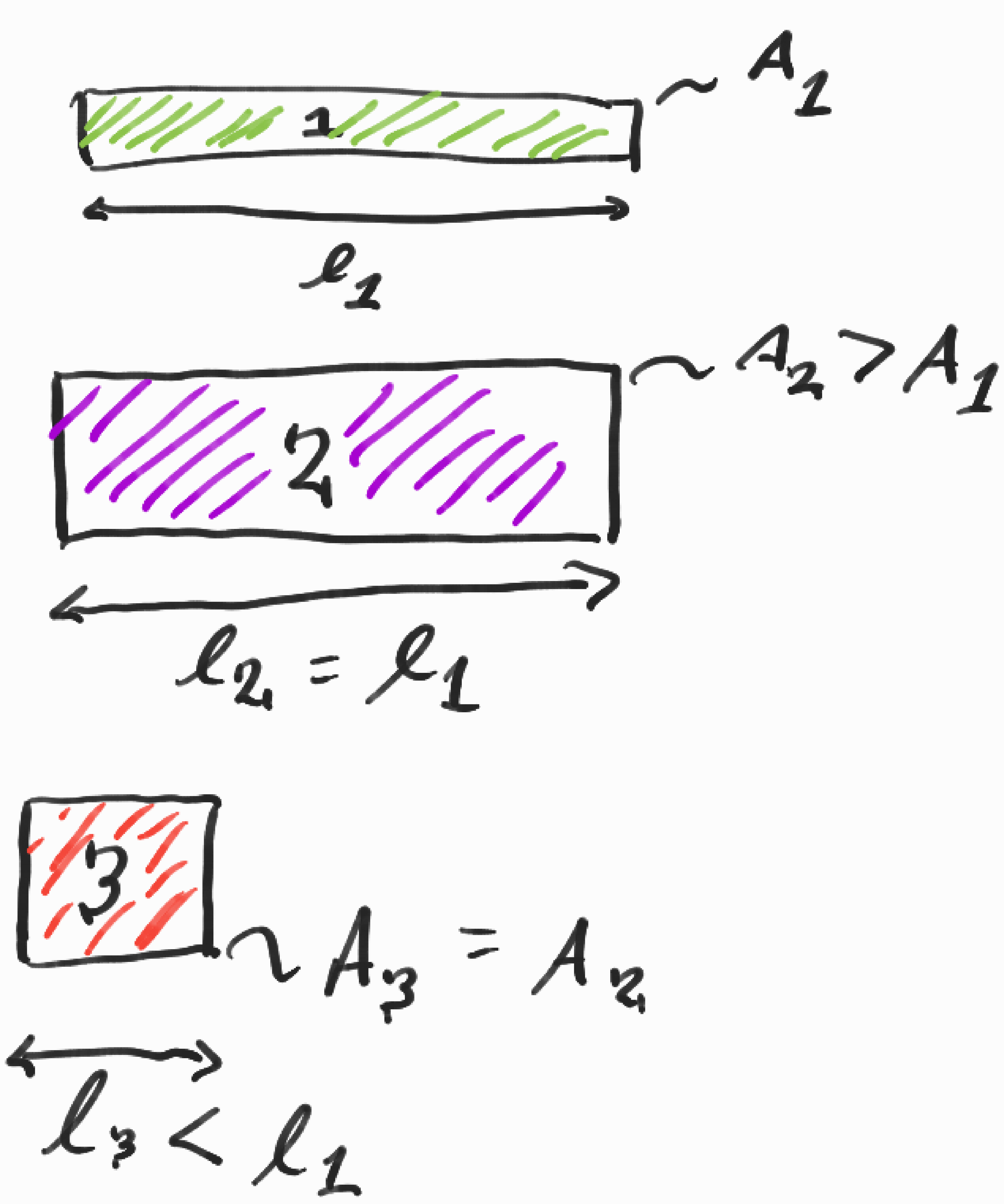
F = force

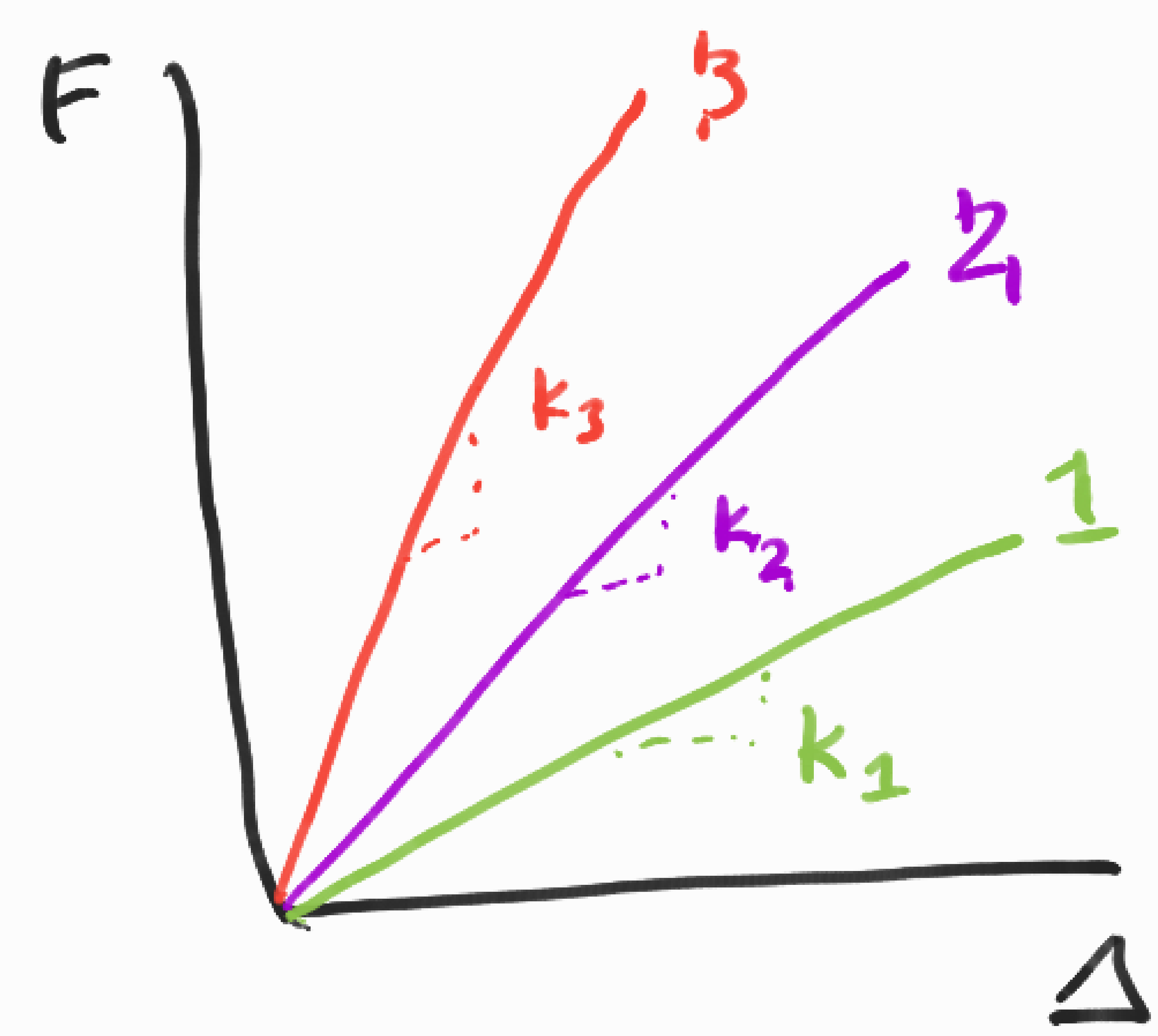
A = cross-section Area

l_0 = undeformed length

Δ = CHANGE IN LENGTH

NOW CONSIDER UNIAXIAL EXTENSION OF 3 BARS OF THE SAME MATERIAL

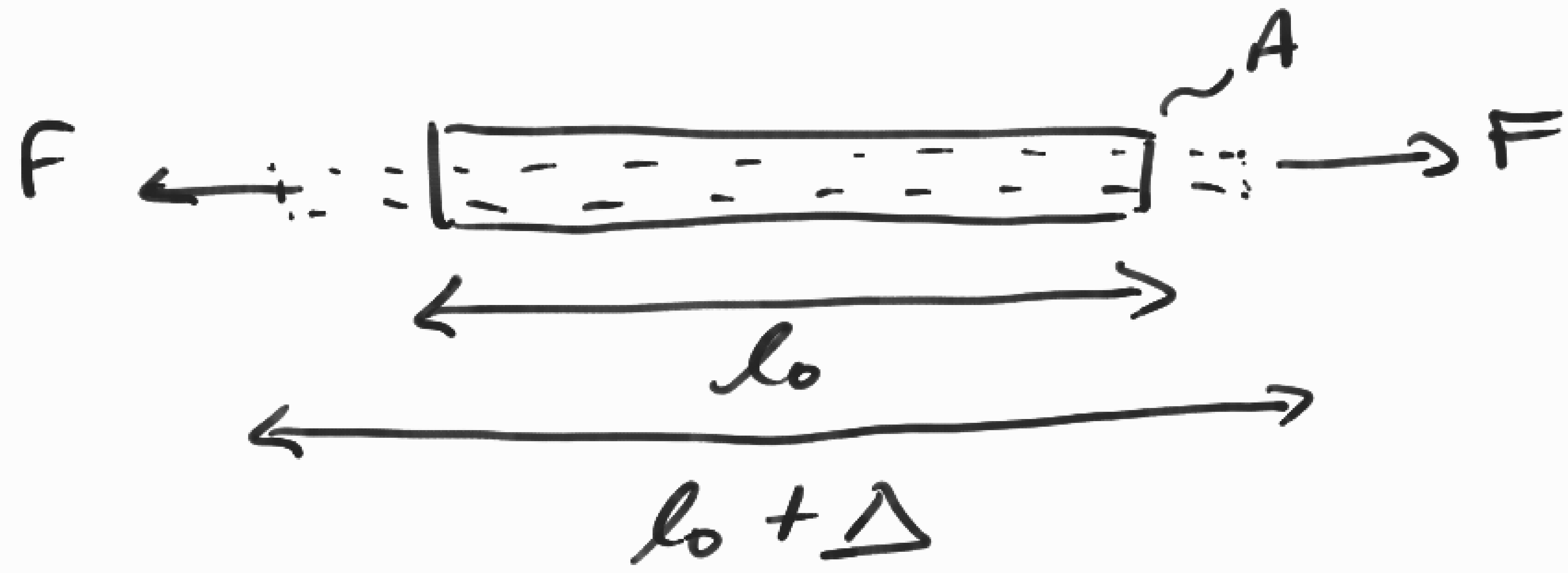




k = STIFFNESS

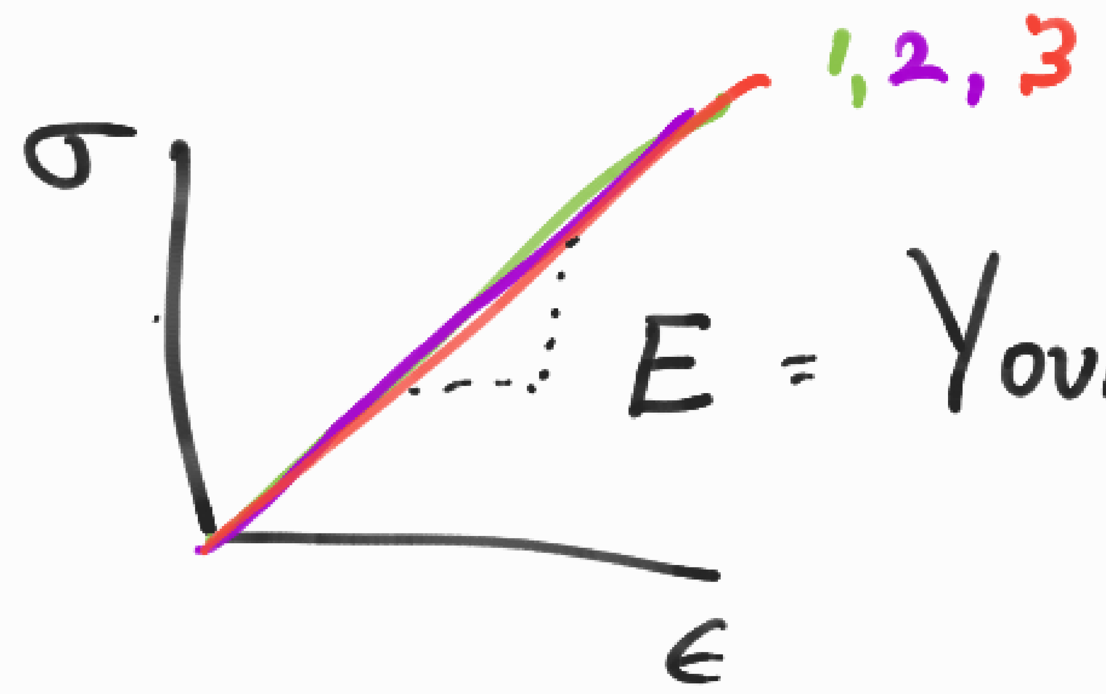
SO STIFFNESS DEPENDS ON GEOMETRY.

WHAT IF WE NORMALIZE:



$$\sigma = \frac{F}{A} \qquad \underline{\underline{4.1}}$$

$$\epsilon = \frac{\Delta}{l_0} \qquad \underline{\underline{4.2}}$$



E = YOUNG'S MODULUS

$$\sigma = E\epsilon \qquad \underline{\underline{4.3}}$$

↳ HOOK'S LAW

HENCE E, WHICH RELATES σ (STRESS) TO ε (STRAIN) IS INDEPENDENT OF GEOM.

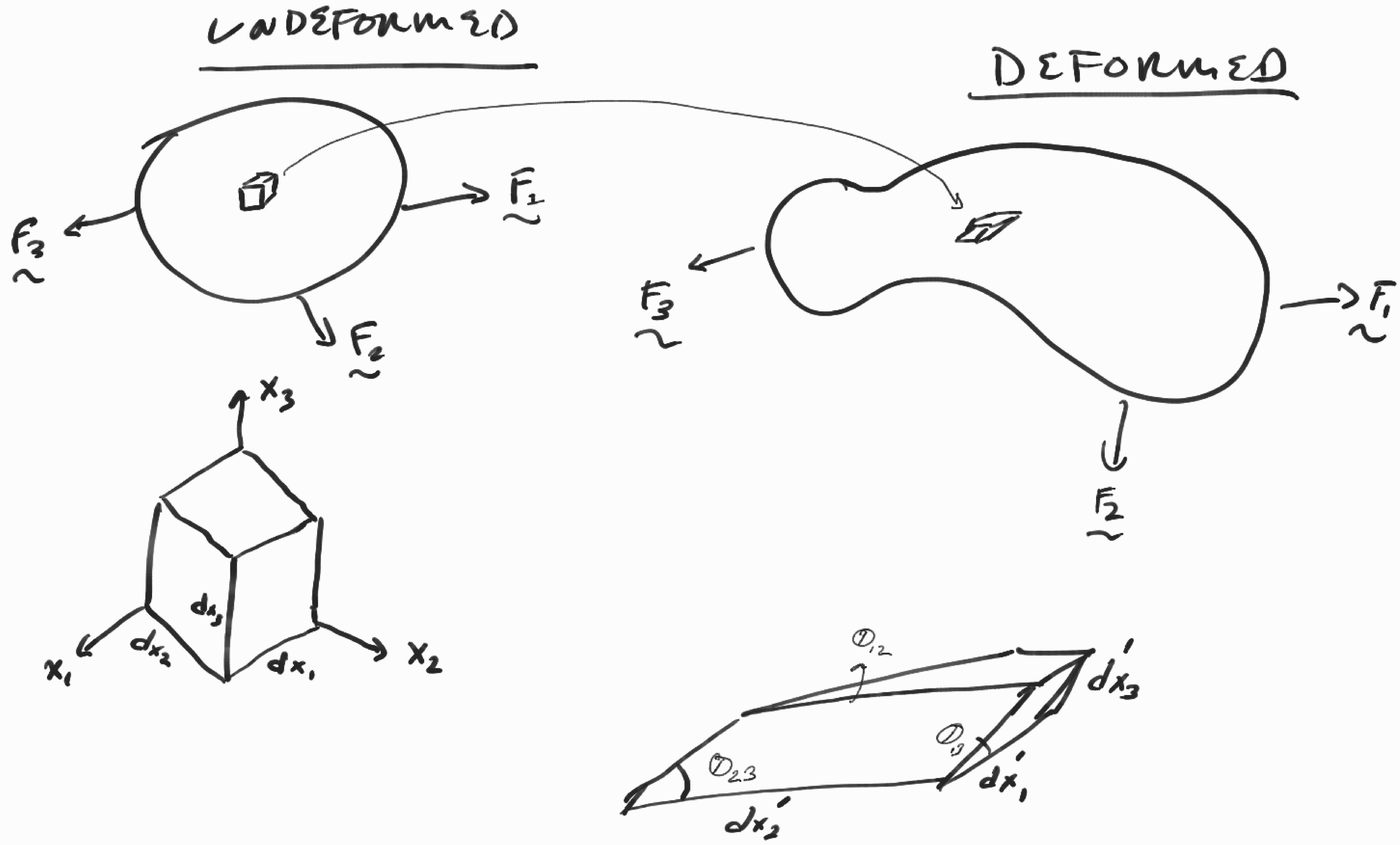
⇒ E IS A "MATERIAL PROPERTY".

4.1 STRAIN

RIGID BODY MOTION - TRANSLATION + ROTATION BUT NO CHANGE IN SHAPE. DISPLACEMENT $\underset{\sim}{u} \neq u(\underset{\sim}{x})$

DEFORMATION - MOTION INVOLVING CHANGE OF SHAPE, HENCE $\underset{\sim}{u} = \underset{\sim}{u}(\underline{x})$ (DISPLACEMENT IS NOT UNIFORM).

CONSIDER A DEFORMABLE BODY SUBJECT TO FORCES F



NORMAL STRAIN (LENGTH CHANGES)

$$\epsilon_{11} = \frac{dx_1' - dx_1}{dx_1} \qquad \epsilon_{22} = \frac{dx_2' - dx_2}{dx_2} \qquad \epsilon_{33} = \frac{dx_3' - dx_3}{dx_3} \qquad \underline{\underline{4.4}}$$

SHEAR STRAIN (ANGLE CHANGE)

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\pi}{2} - \theta_{12}\right) = \epsilon_{21}$$

$$\epsilon_{23} = \frac{1}{2}\left(\frac{\pi}{2} - \theta_{23}\right) = \epsilon_{32} \qquad \underline{\underline{4.5}}$$

$$\epsilon_{13} = \frac{1}{2}\left(\frac{\pi}{2} - \theta_{13}\right) = \epsilon_{31}$$

STRAIN HAS 9 COMPONENTS (4.4, 4.5), EACH ONE IS ASSOCIATED W/ 2 DIRECTIONS $\Rightarrow$ IT IS A TENSOR.

$$\underset{\approx}{\epsilon} = \epsilon_{ij}\, \underset{\sim}{e_i} \otimes \underset{\sim}{e_j} = \epsilon_{11}\, \underset{\sim}{e_1} \otimes \underset{\sim}{e_1} + \epsilon_{12}\, \underset{\sim}{e_1} \otimes \underset{\sim}{e_2} \cdots$$

AND IN MATRIX FORM

off diagonal are shear

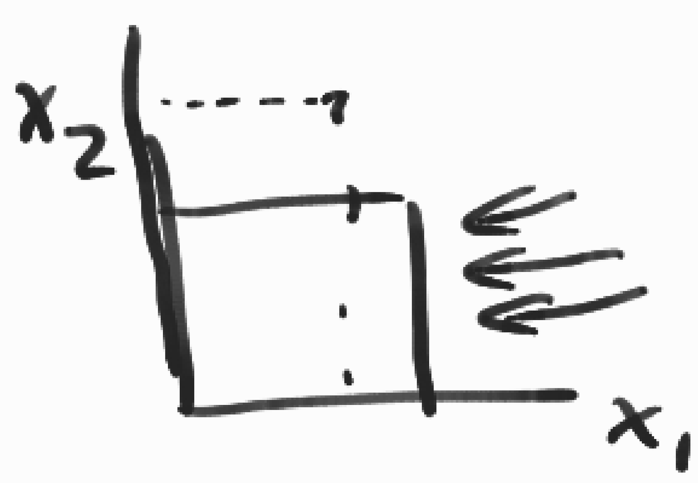
$$[\underset{\approx}{\epsilon}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \qquad \underline{\underline{4.6}}$$

diagonal are normal strains

NOTES

1. ϵ_{ii} (not summed) $> 0 \rightarrow$ extension/stretch

   $< 0 \rightarrow$ compression

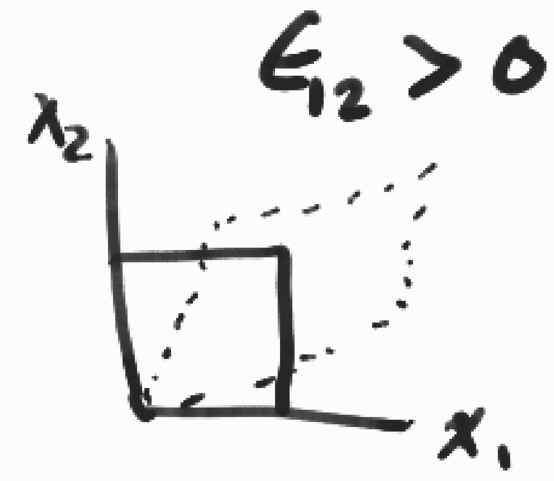


$\epsilon_{11} < 0$

$\epsilon_{22} > 0$

2. ϵ_{ij} $(i \neq j)$ $> 0 \rightarrow$ angle between $i, j \downarrow$

$\epsilon_{12} > 0$

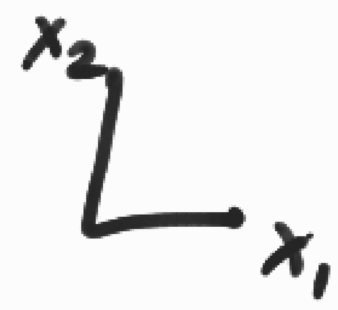


$< 0 \rightarrow$ angle between $i, j \uparrow$

$\epsilon_{12} < 0$

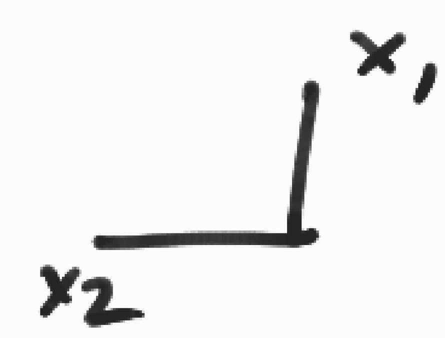
3. $\underset{\approx}{\epsilon}$ DEFINES THE "STATE OF STRAIN" AT A POINT IN A BODY $\Rightarrow$ $\underset{\approx}{\epsilon}$ CAN VARY WITH POSITION

4. ϵ_{ij} ARE COMPONENTS OF $\underset{\approx}{\epsilon}$ IN A SPECIFIC COORD. SYS. HENCE ϵ_{ij} CHANGE W/ COORD SYS, BUT $\underset{\approx}{\epsilon}$ DOES NOT.

FOR 

$\epsilon_{11} > 0$

$\epsilon_{22} < 0$

FOR 

$\epsilon_{11} < 0$

$\epsilon_{22} > 0$

5. STRAIN IS UNITLESS

6. STRAIN IS SYMMETRIC $(\epsilon_{ij} = \epsilon_{ji})$

FINALLY, $\underset{\approx}{\epsilon}$ CAN BE EXPRESSED IN TERMS OF DISPLACEMENT $\underset{\sim}{u}(\underset{\sim}{x})$ USING THE "STRAIN-DISPLACEMENT RELATIONS"

$$\underset{\approx}{\epsilon} = \frac{1}{2}\left(\underset{\sim}{\nabla} \otimes \underset{\sim}{u} + \underset{\sim}{\nabla} \otimes \underset{\sim}{u}^T\right) \qquad \underline{\underline{4.7}}$$

IN CARTESIAN COMPONENT FORM, $\underline{\underline{4.7}} \Rightarrow$

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \qquad \underline{\underline{4.8}}$$

THIS IS 9 EQUATIONS, COMPLETELY EQUIVALENT TO $\underline{\underline{4.4}}-\underline{\underline{4.5}}$

$$\epsilon_{11} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right] = \frac{\partial u_1}{\partial x_1}$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

FINALLY, $\underset{\approx}{\epsilon}$ CAN BE EXPRESSED IN TERMS OF DISPLACEMENT $\underset{\sim}{u}(\underset{\sim}{x})$ USING THE "STRAIN-DISPLACEMENT RELATIONS"

$$\underset{\approx}{\epsilon} = \frac{1}{2}\left(\underset{\sim}{\nabla}\otimes\underset{\sim}{u} + \underset{\sim}{\nabla}\otimes\underset{\sim}{u}^{T}\right) \qquad \underline{\underline{4.7}}$$

IN CARTESIAN COMPONENT FORM, $\underline{4.7}$ =>

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \qquad \underline{\underline{4.8}}$$

THIS IS 9 EQUATIONS, COMPLETELY EQUIVALENT TO $\underline{4.4}$-$\underline{\underline{4.5}}$

$$\epsilon_{11} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right] = \frac{\partial u_1}{\partial x_1}$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

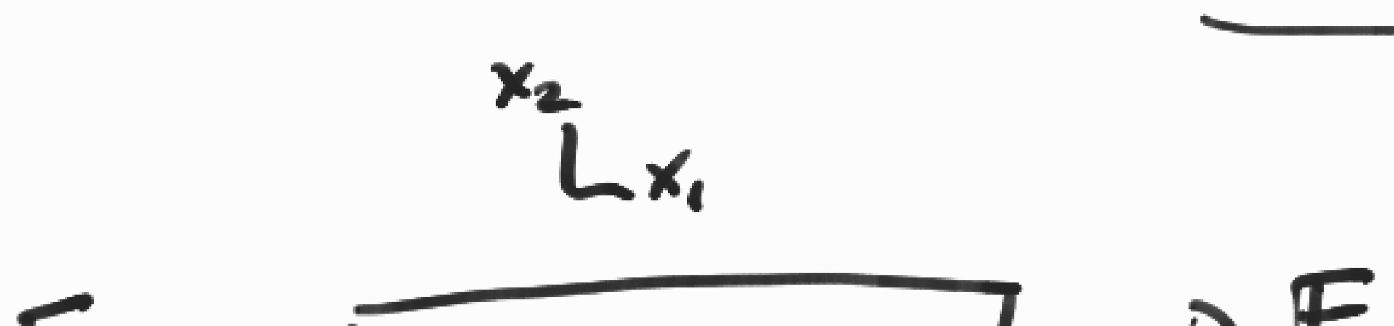
$$\epsilon_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right]$$

$$\vdots$$

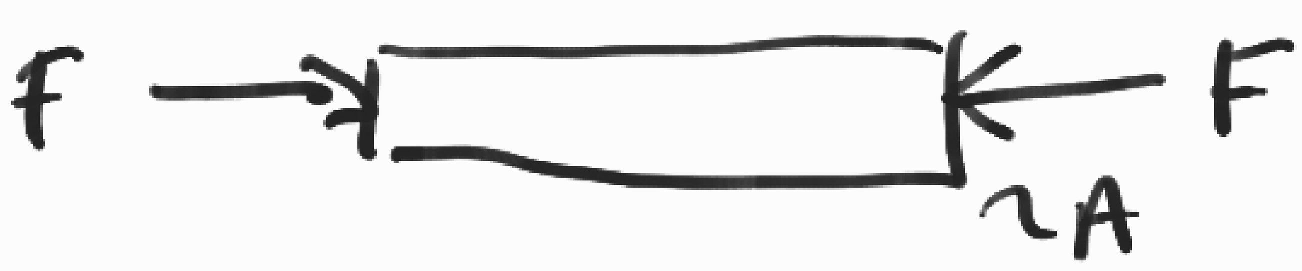
4.2 STRESS ($\underset{\approx}{\sigma}$)

LIKE STRAIN, STRESS $\underset{\approx}{\sigma}$ HAS NORMAL + SHEAR COMPONENTS.

NORMAL



$$\sigma_{11} = \frac{F}{A} \qquad \underline{\underline{4.9}}$$



$$\sigma_{11} = -\frac{F}{A} \qquad \underline{\underline{4.10}}$$

SO $\sigma > 0 \Rightarrow$ TENSION

$\sigma < 0 \Rightarrow$ COMPRESSION

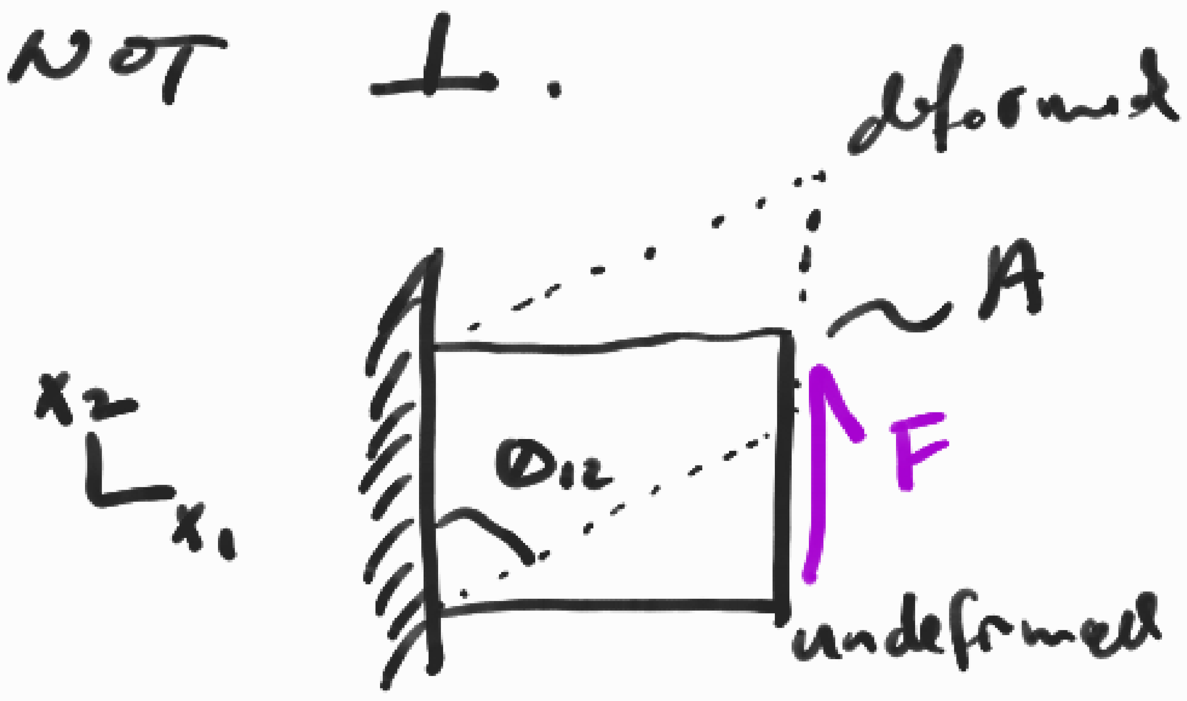
<u>NOTE</u>:

OUTWARD FORCES/STRESSES ARE (+), INWARD ARE (-).

IT IS NOT B/C OF DIRECTION WRT COORD. SYS

SHEAR

IN SHEAR, FORCES ARE PARALLELL TO SURFACE, NOT $\perp$.



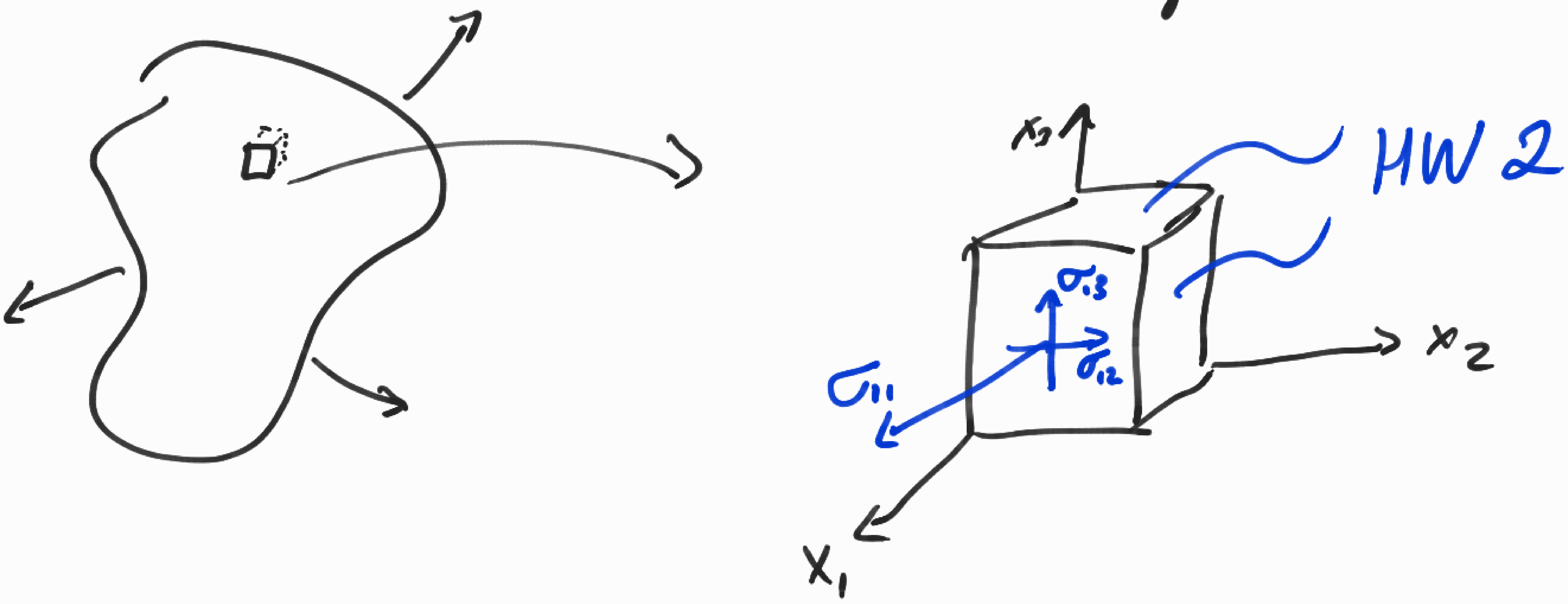
$$\sigma_{12} = \frac{F}{A} \qquad \underline{\underline{4.11}}$$

<u>NOTE</u>: UNLIKE NORMAL STRESS, HERE (+) VS. (-) IS RESULT OF COORD SYS.

$$\rightarrow \sigma_{12} = -\frac{F}{A}$$

$\underline{\underline{4.12}}$

SO IN GENERAL, $\underset{\sim}{\sigma}$ HAS 9 COMPONENTS σ_{ij} WHERE EACH σ_{ij} IS THE STRESS ON A FACE $\perp$ to THE i-DIRECTION, POINTING IN THE j-direction.



WHAT ARE EXAMPLES OF TISSUE LOADED IN TENSION? COMPRESSION? SHEAR?

NOTES: 1) $\underset{\approx}{\sigma}$ IS A 2^{nd} ORDER TENSOR, JUST LIKE $\underset{\approx}{\epsilon}$

2) $\underset{\approx}{\sigma}$ IS SYMMETRIC ($\sigma_{ij} = \sigma_{ji}$)

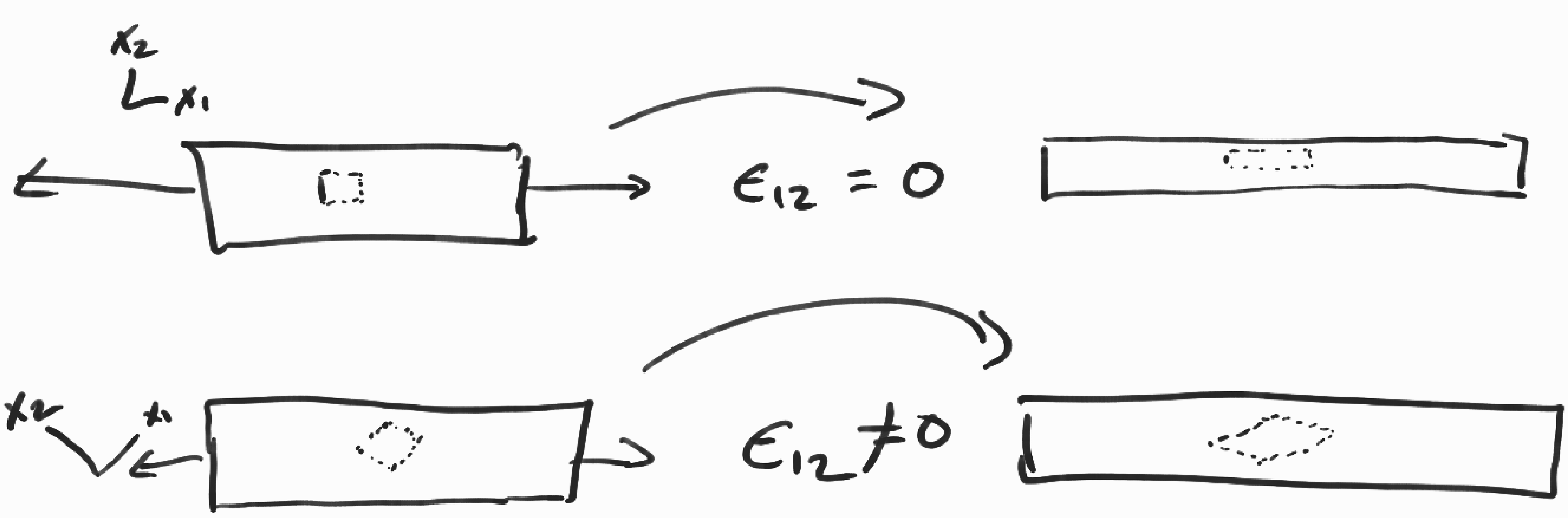
3) σ_{ij} HAS UNITS OF FORCE/AREA, LIKE PRESSURE

$N/m^2 \rightarrow Pa$

4.3 PRINCIPAL STRAIN + STRESS

RECALL $\underset{\approx}{\epsilon}$ AND $\underset{\approx}{\sigma}$ ARE INDEP. OF COOR. SYS

BUT ϵ_{ij} + σ_{ij} ARE NOT.



FOR ANY DEFORMATION, THE COORD SYS CAN BE ROTATED

FOR ANY DEFORMATION, THE COORD. SYS. CAN BE ROTATED TO AN ORIENTATION WHERE $[\underset{\approx}{\epsilon}]$ IS A DIAGONAL MATRIX (i.e. NO SHEAR STRAINS). THE ASSOCIATED BASIS SET $\underset{\sim}{n}_i$ for $i = 1,2,3$ IS CALLED THE <u>PRINCIPAL DIRECTIONS OF STRAIN</u>

THE NORMAL STRAINS ALONG THESE 3 DIRECTIONS, ϵ_i for $i = 1,2,3$, ARE THE PRINCIPAL STRAINS:

Principal strains

$$\underset{\sim}{n}_i \cdot \underset{\approx}{\epsilon} \cdot \underset{\sim}{n}_j = \begin{cases} \epsilon_i & i = j \\ 0 & i \neq j \end{cases} \qquad \underline{\underline{4.13}}$$

Principal directions

HOW DO YOU FIND $\underset{\sim}{n}_i$ AND $\underset{\sim}{\epsilon}_i$? THEY ARE THE EIGEN VECTORS + EIGEN VALUES OF $[\underset{\approx}{\epsilon}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix}$

PRINCIPAL STRESS IS EXACTLY THE SAME, HOWEVER FOR ANISOTROPIC MATERIALS THE PRINCIPAL DIRECTIONS CAN DIFFER FOR $[\underset{\sim}{\sigma}]$ vs $[\underset{\approx}{\epsilon}]$.